Separated Solutions of Almost Periodic Differential Equations

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It is important to look for almost periodic solutions of differential equations since they tend to be the stable ones. In this paper I want to trace two threads of ideas which lead to almost periodic solutions of differential equations because of stability considerations. These two threads converge to give an elegant proof of Bohr's original theorem.

I recall Bohr's original Theorem [1] explicitly. If $F'(x) = f(x)$ and f is almost periodic, then F is almost periodic if and only if F is bounded. This is a theorem about solutions of the differential equation $y' = f(x)$. Although the original theorem was for f complex valued, it holds equally well for f a complex vector function.

The Bohr-Neugebauer Theorem [2] is about the solutions of the equation $x' = Ax + f(t)$ where f is a vector valued almost periodic function and A is a constant matrix. Again a solution x is almost periodic if and only if it is bounded. I will sketch the proof. By a change of variable we may assume that the matrix A is in Jordan canonical form. Then we look at a particular block,

$$x' = \begin{pmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & \lambda \end{pmatrix} x + \hat{f}.$$

The last equation is of the form $x' = \lambda x + f(t)$ with x a scalar. Proceeding up, the rest are of the form $x' = \lambda x + g(t)$ where x is a scalar, and g is a scalar almost periodic function.

If $\mathrm{Re}(\lambda) \neq 0$, then either

$$\int_{-\infty}^{t} e^{\lambda(s-t)} f(s)\,ds \quad \text{or} \quad \int_{t}^{\infty} e^{\lambda(s-t)} f(s)\,ds$$

is a bounded solution which is verified to be almost periodic by an obvious estimate. Since it is the only bounded solution the theorem holds for this component. If $\mathrm{Re}(\lambda) = 0$, then all solutions are bounded and almost periodic if and only if the solution

$$\int_{0}^{t} e^{\lambda(s-t)} f(s)\,ds$$

is bounded and hence almost periodic by Bohr's Theorem. Later we will give a different proof of this result to show how it fits in with other ideas. I remark here that this

Theorem also covers the n^{th} order scalar equation. For this observation one needs to know that for such equations, the solution x being bounded is equivalent to the vector $(x, x', x'', \ldots, x^{(n-1)})$ being bounded.

The Jordan form splits the system into systems living in invariant subspaces. Those coming from $\mathrm{Re}(\lambda) \neq 0$ exhibit the stability features. In those subspaces only one solution is bounded and almost periodic. The rest of the solutions are asymptotic (exponentially) to the almost periodic one either forward in time or backward in time. Equally important, they diverge exponentially either forward in time or backward in time. This situation is called an exponential dichotomy.

Favard attempted to generalize the above to the case when A is an almost periodic matrix. In order to describe his results elegantly we need to introduce some notation. For a real sequence $s = \{s_1, s_2, \ldots\}$ we define $T_s f(t) = \lim_i f(t + s_i)$ whenever this limit exists pointwise. If the limit is to exist in another sense, we will specify each time. We use T_s to denote translation along the sequence s.

The *hull* of an almost periodic function is the collection of functions g such that there is a sequence s for which $g = T_s f$ uniformly. The hull is denoted by $H(f)$ and is compact in $C(-\infty, \infty)$ in the uniform norm. For any $g \in H(f)$ we have $H(g) = H(f)$. Finally for sequences s and s', that s' is a subsequence of s is written as $s' \subset s$.

Along with the equation

$$x' = A(t)x + f(t) \tag{1}$$

we also consider all equations in the *non-homogeneous hull,* namely all equations of the form

$$x' = B(t)x + g(t) \tag{2}$$

where $B = T_s A$ and $g = T_s f$ uniformly, and all equations in the *homogeneous hull*

$$x' = B(t)x. \tag{3}$$

Favard's Theorem [3] is that if for every equation (3) all bounded non-trivial solutions satisfy $\inf_t |x(t)| > 0$, and there is a bounded solution of (1), then each equation (2) has an almost periodic solution.

This theorem includes the Bohr-Neugebauer result since those solutions of $x' = Ax$ which are bounded are almost periodic and therefore do not have zero infimum norm. A quick proof of this for almost periodic solutions of (3) can be given. If $x(s'_i) \to 0$, then take $s \subset s'$ so that $T_s B = C$, $T_{-s} C = B$, $T_s x = y$, $T_{-s} y = x$ all uniformly. It follows that y is a solution of $y' = Cy$ with $y(0) = 0$. Thus $y \equiv 0$ and a fortiori $x \equiv 0$. The simple equation $y'' + y = f(t)$ is an illustration. In phase space (y, y'), the solutions of the homogeneous equation have constant norm, e.g. $|(\cos t, -\sin t)| = 1$.

Favard's proof is to show that every equation in the non-homogeneous hull has a unique solution with minimum norm and that this implies its almost periodicity. We will reproduce this proof below with a modern twist.

Meanwhile we will trace a different thread of ideas which converge with that of Favard. Doss [4] observed that Bohr's Theorem on the primitive of an almost periodic function may be rephrased in the following way. If f is almost periodic then it is easy to see that function $F(t+h) - F(t) = \int_t^{t+h} f(s)ds$ is bounded, continuous and almost periodic. Now the left hand side can be considered without reference to integrals or primitives. He proved then that if F is a bounded continuous function such that for every h the difference $F(t + h) - F(t)$ is almost periodic, then F is almost periodic. It is easy to see that if the hypothesis holds for a dense set of h's and F is uniformly continuous then it holds for all h. A natural question arises, how many differences are required to be almost periodic for this theorem to hold?

Bochner [5] studied a general first order system which includes the possibility of delays and pure difference equations. He showed that if *one* non-trivial difference is almost periodic, and F is bounded and uniformly continuous, then F is almost periodic. In proving this theorem, Bochner derived a new necessary and sufficient condition for a function to be almost periodic. This condition is one which has become very useful in differential equations. *The condition is:* f is almost periodic if it is continuous and if for every pair of sequences t' and s', there are common subsequences (the same choice function for both) t and s such that $T_s T_t f = T_{s+t} f$ pointwise. The meaning of the left hand side is that $T_t f = g$ and $T_s g$ both exist. The usefulness of this criterion is that the condition is *pointwise*. Of course if f is almost periodic these hold uniformly.

If x is a bounded solution to a differential equation $x' = f(t,x)$ where f is almost periodic in t uniformly for x in compact sets, then from every pair of sequences s' and t' one can extract subsequences so that $T_s x = y$, $T_t y$, and $T_{t+s} x$ all exist uniformly on compact subsets of R. By taking further subsequences if neccessary, y will be a solution of the equation $x' = T_s f(t,x)$ and $T_t T_s x$ and $T_{t+s} x$ will both be solutions of the same almost periodic equation $x' = T_{t+s} f(t,x)$. To see how these ideas can be useful we will sketch the proof of Favard's Theorem. Recall that this proof will also prove Bohr's original theorem about primitives being almost periodic if they are bounded.

Sketch of Proof: First, if the set of bounded solutions of equation (2) is non-empty, then this set is a convex set which has a unique element with minimum norm. This is an argument using the parallelogram identity. For two distinct minimizing solutions x and y

$\left|\frac{x+y}{2}(t)\right|^2 + \left|\frac{x-y}{2}(t)\right|^2 = \frac{1}{2}|x(t)|^2 + \frac{1}{2}|y(t)|^2$. Since $\frac{x-y}{2}$ is a bounded solution of the equation (3), the second term is larger than some $\delta > 0$. Taking supremums yields a contradiction. Call the minimum norm solution $x(B,g)$ for (B,g) in the hull of (A,f). If $T_s(A,f) \to (B,g)$ then by taking subsequences if necessary, $T_s x(A,f) = y$ is a solution of

(2) and $\|y\| \leq \|x(A,f)\|$. Now repeat the argument with the sequence $-s$. We get $T_{-s}y$ is a solution of (1) and $\|T_{-s}y\| \leq \|y\| \leq \|x(A,f)\|$. By uniqueness $T_{-s}y = x(A,f)$. It follows that $T_s x(A,f) = x(B,g)$, that is, the least norm solutions are translates of each other. Thus $T_t T_s x(A,f)$ and $T_{t+s} x(A,f)$ are both translates of a least norm solution and solutions of the same equation. By uniqueness they are the same and $x(A,f)$ satisfies Bochner's condition; it is almost periodic.

The above argument can be made with any functional L defined on solutions of equations in the hull of any almost periodic equation provided that 1) each equation has a unique minimizer of the functional L and 2) $L(T_s x) \leq L(x)$ for any solution x. The book of Amerio and Prouse [6] consists of giving examples of weak solutions of partial differential equations which minimize energy functionals. The main difficulty is to prove the existence of a unique minimizer.

A different situation where the Bochner criterion gives an elegant proof of almost periodicity is the case of a unique bounded solution. Specifically, suppose we have a differential equation $x' = f(t,x)$ such that for every equation in the hull, there is only one bounded solution. If x is such a solution, then $T_\alpha T_\beta x$ and $T_{\alpha+\beta} x$ are both the bounded solution of $x' = T_{\alpha+\beta} f(t,x)$ so are equal and the Bochner criterion shows x is almost periodic. It would seem that such a situation is too much to hope for, except there are nice examples where this is true. Moreover, if one replaces the word "bounded", by "with values in a compact set K", then the same argument applies.

It is instructive to consider specifically Bohr's original theorem for a real valued f. Suppose $y' = f(x)$ has the bounded solution $F(x)$. Then the function $G(x) = F(x) - \dfrac{\sup F + \inf F}{2}$ is the solution of the differential equation which is closest to zero in $C(-\infty,\infty)$, and $a \equiv \sup G = -\inf G$. Since it is uniformly continuous, for any sequence s' there is an $s \subset s'$ such that $T_s G$ is a solution of $y' = T_s f$, $-a \leq \inf T_s G$, and $\sup T_s G \leq a$. If strict inequality held, then translation by $-s$ would give a solution of $y' = f(x)$ whose norm is less than a. Consequently, for $K = [-a,a]$ there is a unique solution of each equation $y' = T_s f$ with values in K and G is almost periodic by the above argument. I think this is a very elegant argument.

Solutions which are isolated in a technical sense are called separated; x and y are separated solutions if there is a number d such that $|x(t) - y(t)| \geq d > 0$ for all t. This is the situation in Favard's Theorem. Amerio [7] generalized this to the non-linear case. The hypotheses need apply to all equations in the hull. Suppose that in some compact set K there are only finitely many solutions and that they are separated, then they are all almost periodic.

A property that implies the separated property is uniform stability. Uniform stability is a strong continuity with respect to initial conditions. A solution x is uniformly stable on $[a, \infty)$ if for a given $\varepsilon > 0$ there is a $\delta > 0$ such that if y is a solution such that $|x(t_0) - y(t_0)| < \delta$, then $|x(t) - y(t)| < \varepsilon$ for all $t \geq t_0 \geq a$. Uniform stability of a solution implies that it is separated on intervals of the form $(-\infty, b]$. If $|x-y|\ (0) = \varepsilon$, then $|x-y|\ (t) \geqslant \delta(\varepsilon/2)$ for $t < 0$ else $|x-y|\ (0) < \varepsilon/2$. If t'_n is a sequence such that $T_{t'_n} f = f$ and $t'_n \to -\infty$, take a subsequence $t \subset t'$ such that $T_t x$ and $T_t y$ are solutions of $x' = f(t,x)$. Then $|T_t x - T_t y| \geqslant \delta\,(\frac{\varepsilon}{2})$ on the real line.

The various concepts of stability and the relationships with almost periodicity have been studied intensly by Seifert, Yoshizawa, Fink and others including researchers in the USSR. The book by Levitan and Zhikov [8] outlines these developments in the USSR, while Fink [9] discusses all of the above ideas. A more complete discussion of all aspects of stability and almost periodicity is given in Yoshizawa [10].

Some specific equations to which the above ideas apply can also be found in [9]. One of the more remarkable results is that of Frederickson and Laser [11]. The equation $x'' + f(x)\ x' + x = p(t)$ with almost periodic p has an almost periodic solution if and only if $F(\infty) - F(-\infty) > \pi\beta$ where $F(x) = \int_0^x f$ and $\beta = \max_s M\{p(t) \sin (s-t)\}$. The solution is uniformly quasi-asymptotically stable in the large.

For scalar equations $x' = f(x,t)$, if there is a bounded uniformly stable solution on $[0,\infty)$ then there is an almost periodic solution. If f is monotone in x and there is a bounded solution on $[0,\infty)$ then there is an almost periodic solution. Each of these solutions is a unique solution contained in some compact set. The compact set is obtained by separation in the first case and by minimization of the oscillation function in the second.

A second order example is the equation $x'' = m(x)x' + g(x) + e(t)$ where e is almost periodic. Define $M(x) = \int_0^x m$ and

$$\phi(u,v) = \begin{cases} \dfrac{M(u+v)-M(u)}{v} & v \neq 0; \\ m(u) & v = 0; \end{cases} \qquad h(u,v) = \begin{cases} \dfrac{g'(u+v)-g'(u)}{v} & v \neq 0. \\ g'(u) & v = 0. \end{cases}$$

If there is an $a < b$ for which

$$g(a) + e(t) \leqslant 0 \leqslant g(b) + e(t) \text{ holds for all } t,\ g'(t) > 0,$$

and there is a λ so that

$$(\phi(u,v) - \lambda)^2 - 4h(u,v) \leqslant 0 \text{ for } u,\ u+v \in [a,b],$$

then there is an almost periodic solution. This is a unique solution with values in $[a,b]$ and is stable.

A slightly different set of sufficient conditions is illustrated by the equation

$$x'' + f(x)\ x' + g(x) = k\ p(t)$$

where p is almost periodic. Let $F(x) = \int_0^x f$, $g(0) = 0$, g' exist and satisfy $0 < g'(x) < \beta$ and $f(x) \geqslant \alpha$ where $\beta < \alpha^2$. Suppose one can find $c < d$ so that $g(c) = -k$ and $g(d) = k$ and $a < b$ such that $k < \min\{\,[F(d) - F(c)]f(x) + g(-b), [F(-a) - F(-b]f(x) - g(d)\,\}$ on $[-b, d]$, then there is a unique bounded solution which is uniformly stable and almost periodic.

Finally, I mention that the notions of stability and separatedness have their counterparts in the abstract theory of dynamical systems. The use of dynamical systems for non-autonomous equations was inaugurated by Miller [12].

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